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The oriented cycle game

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Abstract

Two players A and C play the following game on a graph G. They orient the edges of G alternately with C playing first until all the edges of G have been oriented. The goal of C is to create at least one oriented cycle, while A wants to avoid this and finish with an acyclic orientation.

Among other results we determine the minimal integer $m = m(n)$ such that C has a winning strategy on every graph of order n and size m. We also discuss several generalizations of this game. © 1998 Elsevier Science B.V. All rights reserved

1. Introduction

In a recent paper of Chartrand et al. [7] the following game is considered. Two players orient the edges of a graph alternately: one player wants to achieve a strong orientation and the other aims to avoid this. Note that the game described in the abstract is more favorable for C, as every strong orientation contains an oriented cycle as well. However the oriented cycle game can be interesting on a larger class of graphs, as they do not need to be 2-connected.

Another similar game is examined by Aigner et al. [1] and Alon and Tuza [4]. In that game one player wants to find an unknown acyclic orientation of a graph by asking the orientation of certain edges from the other player. The authors estimate the number of questions the first player needs to find the acyclic orientation and study the class of exhaustive graphs, i.e. graphs where all edges must be asked.

The oriented cycle game is a variant of the previous two games: two players A and C orient the edges of a graph G alternately with C playing first until all edges have been oriented. Player C (creator) wins if the resulting digraph has at least one oriented cycle, A wins if he could avoid this so the orientation remained acyclic.

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The winner of the oriented cycle game can be determined easily in some special graphs. Looking at complete graphs first, player A clearly wins on K_3 but player C wins on K_4 and any larger complete graph. Moreover,

the creator C wins the oriented cycle game on every G containing K_4.

Although this follows from the more general Theorem 3.3, we suggest that the reader verifies the assertion by following the possible steps of the two players, to get a better feeling about the game.

In Section 2 we characterize the chordal graphs where C wins the oriented cycle game exactly by containing K_4 as a subgraph. We also show that A wins the game in every outerplanar graph using the same method.

Generally it is not true that if C can win the oriented cycle game on H then C can win on every G containing H, because playing outside of H first could change the order of the two players in H and possibly changing the result of the game (see Section 5 for an example), however the following is clearly true:

if C can win the game on H, G contains H and $G - H$ has an even number of edges then C wins on G as well.

The winning strategy for C can be to start in H and play in H whenever A does so, and play in $G - H$ whenever A does so. The parity of the outside edges ensure that two players are essentially playing the oriented cycle game on H where C can create an oriented cycle winning the game on G as well.

If we try to maximize the size of graphs of order n where A wins, we find several examples for graphs with $2n - 3$ edges where A wins. In Section 3 we show that this size is the most we can get in any graph, i.e. playing on any graph of order n and size at least $2n - 2$ player C wins the game. Using random methods this implies the existence of graphs with arbitrary large girth where C wins the game.

A stronger version of the oriented cycle game is examined in Section 4. We call the following version the *rotor game*: suppose the two players orient the edges of a (V_1, V_2, V_3) tripartite graph (or in general k-partite with $k \geqslant 2$) with $|V_1| = |V_2| = |V_3|$. The goal of C is to achieve complete oriented matchings from V_1 to V_2, from V_2 to V_3 and from V_3 to V_1, while A tries to block this. Clearly if C can win the rotor game on a graph, he also wins the oriented cycle game as there will be at least one cycle formed by edges of the matchings.

In the last section we mention some possible generalizations of the oriented cycle game and a couple of open problems as well.

Throughout the paper let $\delta_H(G)$ denote the minimal degree of vertices of G in H and for any $U \subset V(H)$, $H[U]$ the induced subgraph of H spanned by U, as usual. We also use several times the following estimates for the binomial coefficients:

$$\binom{n}{k} \leqslant \left(\frac{en}{k}\right)^k,$$

$$e^{-1/6n}(p^p q^q)^{-n}/\sqrt{2\pi pqn} \leqslant \binom{n}{k} \leqslant (p^p q^q)^{-n}/\sqrt{2\pi pqn}$$

if $1 \leqslant k = pn \leqslant n/2$ and $q = 1 - p$. Both inequalities can be derived easily from Stirling's formula (see, e.g., [6, p. 4]).

2. Chordal and outerplanar graphs

Theorem 2.1. *If G is a chordal graph, the creator C wins the oriented cycle game if and only if G contains K_4.*

Proof. We have seen that C can win if G contains a K_4. On the other hand suppose G is chordal and does not contain K_4. We prove more than we need: A wins the oriented cycle game, even if C is allowed to 'pass', not orienting any edge, forcing A to play two (or more) consecutive turns.

We apply induction on the order of G. If G has at most 3 vertices the assumption is trivially true. If G is not 2-connected, let A follow the strategy of playing in the same 2-connected component where C played last whenever it is possible according to his winning strategy in that component. If C completes the orientation of a 2-connected component, A plays in any other block (this corresponds to the pass of C). In any 2-connected component the oriented cycle game is played with C passing sometimes resulting an acyclic orientation of G by induction. From now on we suppose that G is 2-connected.

First we show that there is a vertex v of degree 2. Let H be the hypergraph obtained from G by assigning a vertex to each triangle of G, and forming an edge of those triangles that share a particular edge of G (only if there are more than one, so we do not get singleton edges). Clearly, any vertex of H has degree at most three and at most one edge goes through any two vertices. Our conditions on G imply that H is acyclic. Indeed, if H contains a cycle, the corresponding triangles in G form a cycle. Hence, as G is chordal, G contains a K_4. Now, as H is acyclic, it has a vertex used in at most one edge, so the corresponding triangle has only one edge used by another triangle. The vertex opposite to this edge has degree 2.

Let x and y be the neighbors of v and set $G' = G - \{v\}$. Note that G' is also a K_4-free chordal graph so, by the induction hypothesis, A wins the game on G'. A winning strategy for A on G is as follows. He playes in G' whenever C does, following his winning strategy in G' and if C orients an edge incident with v, A replies by orienting the same way the other edge incident with v. Thus v becomes a sink or a source and cannot be used in any oriented cycle.

Note that C can force A to play in $G - G'$ first if all edges of G' are already oriented, but as this orientation is acyclic, we can suppose without loss of generality that there is no directed path from y to x. Thus A can orient $\overrightarrow{xv}$, and even if C finishes with $\overrightarrow{vy}$ and xvy becomes an oriented path, it will not produce an oriented cycle. With this strategy A clearly wins the game on G. $\square$

Theorem 2.2. *A wins the oriented cycle game on every outerplanar graph G.*

Proof. The proof is similar to the previous one, with C passing again whenever he wants. We apply induction on the number of vertices. Player A clearly wins if the graph has at most 3 vertices.

Suppose there exists $v \in V(G)$ of degree 1. Clearly A can play the game according to his winning strategy on $G' = G - \{v\}$ and the orientation of the only edge incident with v does not produce any oriented cycle no matter who orients it.

If there is no vertex of degree 1, then there must exist $v \in V(G)$ of degree 2 adjacent to only $x, y \in V(G)$ (we can take a vertex of an end-region for example). If $\{xy\} \in E(G)$ then setting $G' = G - \{v\}$, as before, completes the proof the same way as in Theorem 2.1. Otherwise add the edge $\{xy\}$ to G' keeping it outerplanar and finish the proof the same way as before. □

It would be interesting to characterize other classes of graphs (for example planar or bipartite graphs) with respect to the winner of the oriented cycle game.

3. Edge density

In this section we show that whenever a graph G with n vertices has at least $2n-2$ edges (or it has a subgraph like that with $n \geqslant 3$), the creator C wins the oriented cycle game on G. This is best possible as one can find several families of graphs with $2n-3$ edges on which player A can win. For example every maximal outerplanar (or K_4-free chordal) graph on n vertices has $2n-3$ edges and, by Theorem 2.2 (or 2.1), A wins on these graphs. An infinite family of bipartite graphs of size $2n-3$ on which A wins the game can be obtained by adding vertices of degree two to $K_{3,3}$ (see Section 5).

Let $G \subset H$ be multigraphs with $|G| = n$. A *k-system of G in H* is a set of kn edges of H, forming n stars of k edges each, with exactly one star centered at each vertex of G. In particular a *2-system* is a collection of n edge disjoint 2-trails, one centered at each vertex of G. A *quasi k-system in H* is a k-system from which one edge is missing. Thus a *quasi 2-system* has $n-1$ edge disjoint 2-trails centered at different vertices of G and an additional edge adjacent to the remaining vertex. Clearly H must have at least kn edges to contain a k-system on n vertices and $kn-1$ edges to contain an quasi k-system. From now on we are interested in 2-systems and quasi 2-systems only, although the following results could easily be generalized for k-systems as well.

In [8] Tarsi showed the existence of certain decompositions of graphs into stars. We need a variant of this result, the proof of which is based on the idea used in [3,8].

Lemma 3.1. *Let H be a multigraph with $k+1$ vertices and $2k-1$ edges, with no loop at a special vertex $v \in V(H)$, and let $G = H - \{v\}$. Suppose that G is balanced, i.e. for any $L \subset V(G)$, $|L| = l$ the subgraph $H[L \cup \{v\}]$ has at most $2l$ edges. Then G has a quasi 2-system in H.*

Proof. Let F be the $(2k-1, 2k)$ bipartite graph on the classes of vertices A and B, where $A = E(H)$ and B is the union of two copies of each vertex $w \in V(G)$. Each member $e=\{u,w\}$ of $E(H)$ is joined by edges in F to the two copies of u and w in B.

The desired quasi 2-system clearly exists if we can find a matching of size $|A|=2k-1$ in F as all but one vertex of G will get two disjoint edges adjacent with it while only one edge remains for the last vertex. All we need to check is that Hall's condition is satisfied in F to ensure this matching.

Let $E' \subset E(H)=A$ be a set of edges of H intersecting the vertices of G in a set L. With $|L|=l$ E' must have $2l$ neighbours in B. On the other hand the set $L\cup\{v\}$ spans at most $2l$ edges in H so $|\Gamma(E')|=2l\geqslant|E'|$. Thus there is a desired matching and so G contains a quasi 2-system. □

Lemma 3.2. *Let H be a multigraph with $k+1$ vertices and $2k$ edges, with no loop at a special vertex $v\in V(H)$, and let $G=H-\{v\}$. Suppose that G is almost balanced: for any $L\subset V(G)$, $|L|=l$ the subgraph $H[L\cup\{v\}]$ has at most $2l+1$ edges. Then G has a quasi 2-system in H.*

Proof. We proceed as in the previous proof. Let $F(A,B)$ be the $(2k,2k)$ bipartite graph defined as before. Again we need a matching of size $2k-1$ to get the quasi 2-system. This time we will use a slight generalization of Hall's theorem: if in a bipartite graph $G(A,B)$ each $A'\subset A$ subset has at least $|A'|-1$ neighbours in B then there exists a matching of size $|A|-1$ (see, e.g., [5, p. 56]).

In fact if $E'\subset E(H)=A$ intersects the vertices of G in a set L where $|L|=l$, E' will have $2l$ neighbours in B. By our condition the set $L\cup\{v\}$ spans at most $2l+1$ edges in H so $|\Gamma(E')|=2l\geqslant|E'|-1$. Thus there is a matching and so G contains a quasi 2-system. □

Theorem 3.3. *If a graph G has a subgraph H of order n and size at least $2n-2$ $(n\geqslant 3)$ then the creator C wins the oriented cycle game on G.*

Proof. Let H be a minimal subgraph of order n (at least 3) and size at least $2n-2$. We shall show that C can achieve an oriented cycle in this subgraph with the following strategy.

The creator C starts the game by orienting an arbitrary edge $\overrightarrow{vu}$ of H. The strategy of C will be to ensure that no out-degree (or in-degree) in H will be 0 at the end of the game, hence H has no sink (or source) so its orientation cannot be acyclic. To achieve this C will use certain 2-systems or quasi 2-systems.

Suppose first that A orients an edge outside of H. Then $H'=H-uv$ has n vertices and at least $2n-3$ edges. In $J=H'-\{v\}$ any $L\subset V(J)$, $|L|=l<n-1$ forms at most $2l$ edges in $H'[L\cup\{v\}$ by the minimality of H. By Lemma 3.1 J has a quasi 2-system of disjoint 2-trails centered at each vertex of J except $z\in J$, which has only one edge corresponding. The creator C continues by orienting this edge away from z. Thus v and z cannot be sinks in H as they have out-degree 1 after two steps. From now on whenever A orients an edge xy of H that was used as a xyt 2-trail in the quasi

2-system, C always orients yt away from y, making the out-degree of y positive. If A plays outside H or orients an edge that was not used in the 2-system, C is free to choose any edge of the 2-system and orient it away from the corresponding central vertex. Clearly with this strategy the oriented subgraph H has no sink so must contain an oriented cycle at the end, hence C wins the game.

If the first move of A is in H then without loss of generality we may assume that it is orienting an edge $\overrightarrow{wt}$ such that $v \neq w$ (otherwise $u \neq t$ and we make all in-degrees positive the same way). Thus, after two steps, v and w have out-degree 1. Just like above, it is sufficient to show that after deleting v and w the remaining part of H has a quasi 2-system, and so C wins the game with the exact same strategy. We need to consider two cases.

Suppose first that $\{vw\} \in E(H)$. Let H' denote the graph obtained from H by deleting the edges vu, wt and vw and contracting v and w. Then H' has $n-1$ vertices, at least $2n-5$ edges, and the subgraph $J = H' - \{v = w\}$ is balanced as any $L \subset V(J)$, $|L| = l < n-2$ spans at most $2l$ edges in $H'[L \cup \{v = w\}$, otherwise $H[L \cup \{v, w\}]$ had $l+2$ vertices and at least $2l+2$ edges contradicting the minimality of H. Again by Lemma 3.1 J has a quasi 2-system.

Similarly, if $\{vw\} \notin E(H)$ then denote by H' the graph obtained from H by deleting the edges vu and wt and contracting v and w again. This H' has $n-1$ vertices, and at least $2n-4$ edges. To check the density condition, set $J = H' - \{v = w\}$ and note that any $L \subset V(J)$, $|L| = l < n-2$ spans at most $2l+1$ edges in $H'[L \cup \{v, w\}$. Indeed, otherwise $H[L \cup \{v, w\}]$ has $l+2$ vertices and at least $2l+2$ edges, contradicting the minimality of H. Now by Lemma 3.2 J has a quasi 2-system. This completes the proof. □

As there exist graphs of large girth and large size (see e.g., [6, Corollary 19, p. 53]) the following follows easily.

Corollary 3.4. *Given $g \geqslant 3$, if n is sufficiently large then there is a bipartite graph G of order n, size $2n-2$ and girth at least g on which the creator C wins the oriented cycle game.*

4. Rotor game

The results below could be proved in a good many different ways by using a variety of random graphs: random regular graphs, random bipartite graphs obtained as unions of complete matchings, random graphs obtained from $G_{k\text{-out}}$ or random graphs obtained from $G_{k\text{-reg}}$. Here we shall opt for the last approach as we do not try to minimize k but rather go for simpler calculations.

Let $\mathscr{G}(n, n; k\text{-reg}, k\text{-reg})$ be the class of random graphs obtained as follows. Let U and W be disjoint sets, each with n elements. Construct a random red-blue coloured bipartite multigraph with bipartition (U, W) in the following way. Join each $u \in U$ to W by k red edges, say $e_1(u), e_2(u), \ldots, e_k(u)$, with $\mathbb{P}(e_i(u) = uw) = 1/n$ for each i and each $w \in W$; and similarly, join each $w \in W$ to U by k blue edges, say $f_1(w), f_2(w), \ldots, f_k(w)$, with $\mathbb{P}(f_i(w) = uw) = 1/n$ for each i and each $w \in W$. Thus for every $u \in U$ we pick k elements of W independently, with replacement, and colour the edges red; the blue edges are obtained analogously. All choices are independent. A typical element of $\mathscr{G}(n,n; k\text{-reg}, k\text{-reg})$ is the union of these red and blue graphs.

Note that each $G \in \mathscr{G}(n,n; k\text{-reg}, k\text{-reg})$ has kn red and kn blue edges; every vertex in U has red degree k and every vertex in W has blue degree k.

Lemma 4.1. *If $k \geqslant 8$ then a.e. $G \in \mathscr{G}(n,n; 2k\text{-reg}, 2k\text{-reg})$ is such that if $A \subset U$, $B \subset W$, $|B| < |A|$, $|A| + |B| \leqslant n$ and every vertex of A is joined to B by at least k red edges then $|A| \geqslant n/6$.*

Proof. Let $b = \beta n < a = \alpha n \leqslant n/4$ and write $E(a,b)$ for the expected number of full pairs (A,B), i.e. pairs (A,B) such that $A \subset U$, $|A| = a$, $B \subset W$, $|B| = b$ and every vertex in A is joined to B by at least k red edges. Clearly

$$E(a,b) = \binom{n}{a}\binom{n}{b}\left\{\sum_{l=k}^{2k}\binom{2k}{l}\left(\frac{b}{n}\right)^{l}\left(1-\frac{b}{n}\right)^{2k-l}\right\}^{a}$$

$$\leqslant \binom{n}{a}\binom{n}{b}\left\{2\binom{2k}{k}\left(\frac{b}{n}\right)^{k}\left(1-\frac{b}{n}\right)^{k}\right\}^{a} = E'(a,b).$$

Note that

$$\frac{E'(a,b-1)}{E'(a,b)} \leqslant \frac{b}{n-b+1} < \frac{1}{2},$$

so

$$\sum_{b<a\leqslant n/4} E(a,b) \leqslant \sum_{a\leqslant n/4} E'(a,a).$$

Also for $1 < a \leqslant n/6$,

$$E'(a,a) \leqslant \left(\frac{en}{a}\right)^{2a}\left\{2^{2k+1}\left(\frac{a}{n}\right)^{k} e^{-ak/n}\right\}^{a} \leqslant \left\{e^2 2^{2k+1}\left(\frac{a}{n}\right)^{k-2}\right\}^{a}$$

$$\leqslant \{2e^2 4^k 6^{-(k-2)}\}^a < \{500(2/3)^k\}^a < (5/6)^a,$$

and also if $k+1 \leqslant a \leqslant n^{1/2}$ then

$$E'(a,a) \leqslant \left\{e^2 2^{2k+1}\left(\frac{a}{n}\right)^{k-2}\right\}^{a} \leqslant n^{-a/3}.$$

Thus

$$\sum_{b<a\leqslant n/4} E(a,b) \leqslant \sum_{a\leqslant n/4} E'(a,a) \leqslant \sum_{a=k+1}^{\infty} n^{-a/3} + \sum_{a=n^{1/2}}^{n/6} (5/6)^a$$

$$\leqslant n^{-k/3} n^{-1/3} + \tfrac{1}{6} n (5/6)^{n^{1/2}} < n^{-k/3}$$

so almost no $G \in \mathscr{G}(n,n;2k\text{-reg},2k\text{-reg})$ contains a full pair (A,B) with $|B| < |A| \leqslant n/6$. $\square$

Lemma 4.2. *For $k \geqslant 2^8$ a.e. $G \in \mathscr{G}(n,n;8k\text{-reg},8k\text{-reg})$ is such that for every set $A \subset U$, $|A| \geqslant n/6$, at most $n/4$ vertices in W send fewer than k blue edges to A.*

Proof. Fix $A \subset U$, $|A| = a \geqslant n/6$. The probability that a vertex $w \in W$ sends at most $k-1$ blue edges to A is

$$p = \sum_{l=0}^{k-1} \binom{8k}{l} \left(\frac{a}{n}\right)^l \left(1-\frac{a}{n}\right)^{8k-l} < \binom{8k}{k} \left(\frac{a}{n}\right)^k \left(1-\frac{a}{n}\right)^{7k}$$

$$< 8^k (8/7)^{7k} (1/6)^k (5/6)^{7k} = \{(4/3)(20/21)^7\}^k < 3^{-8}.$$

Hence the probability that the conclusion of the lemma fails for a random graph $G \in \mathscr{G}(n,n;8k\text{-reg},8k\text{-reg})$ is no more than

$$2^n \sum_{l>n/4} \binom{n}{l} p^l < 2^n \sum_{l>n/4} \binom{n}{l} 3^{-8l} < 2^{n+1} \binom{n}{n/4} 3^{-2n} < 2^{-n}. \qquad \square$$

Now suppose that the two players M and N are playing the following game on a bipartite graph G with bipartition (U,W). They orient the edges of the graph alternately like in the oriented cycle game. But this time the goal of M is to achieve a complete oriented matching from U to W, while N tries to block this. We call this game the *complete matching game.*

Theorem 4.3. *For $k \geqslant 2^8$ the constructor M wins the complete matching game on a.e. $G \in \mathscr{G}(n,n;8k\text{-reg},8k\text{-reg})$.*

Proof. We know that a.e. $G \in \mathscr{G}(n,n;8k\text{-reg},8k\text{-reg})$ satisfies the conclusions of Lemmas 4.1 and 4.2 with the original partition (U,W) and also with U and W interchanged. We claim that M has a winning strategy on every such G. Obviously M should orient all edges from U to W, while N should orient all edges backwards, so the only question is what is the optimal order to choose the edges.

The winning strategy for M is of the simpliest kind: M follows the lead of N at random. To be precise, if N chooses a red edge incident with a vertex $u \in U$, M does the same at random: he picks at random one of the remaining red edges incident with u. If there is no such edge, M makes a random move. Similarly, if N chooses a blue edge incident with a vertex $w \in W$, M does the same, at random, if such a move is possible, otherwise he makes a move at random.

To prove the theorem it suffices to show that, no matter what strategy N follows, the probability that these random moves result in a win for M is strictly positive. In fact, we shall prove more; namely that the probability that M wins in this way when playing against the optimal strategy of N tends to 1.

Let H be the graph formed by the edges chosen by M. What is the probability that, no matter what N does, H fails to contain a complete matching?

Suppose H does not have a complete matching. By Hall's theorem, there is a set $A \subset U$ such that $|\Gamma_H(A)| < |A|$. Since with $A' = W - \Gamma_H(A)$ we have $\Gamma_H(A') \subset U - A$, we also have $|\Gamma_H(A')| < |A'|$ and $|\Gamma_H(A)| + |A| + |\Gamma_H(A')| + |A'| \leqslant 2n$. Hence, for the loss of a factor 2 in the probability, we may assume that an obstruction $|\Gamma_H(A)| < |A|$ to the complete matching happens at a set $A \subset U$ with $|\Gamma_H(A)| + |A| \leqslant n$.

Let us bound the probability that such an obstruction happens at a set $A \subset U$ with $|A| = a$. By Lemma 4.1 $a \geqslant n/6$ and by Lemma 4.2 in G at most $n/4$ vertices in W send fewer than k blue edges to A. Let W' be a set of vertices sending at least k blue edges to A, with $|W'| = 3n/4$. What is the probability that a fixed vertex $w \in W'$ does not belong to $\Gamma_H(A)$? If $w \notin \Gamma_H(A)$ then at no time does M pick a $w - A$ edge. The first time M picks a blue edge incident with w, the probability of picking an edge leading to A is at least $(k-1)/(8k-1)$: either at least k of $8k$ blue edges lead to A or at least $k-1$ of the remaining $8k-1$ blue edges do so after N has picked a $w - A$ edge. If the first blue edge incident with w picked by M does not lead to A, then the probability that the second edge will do so is at least $(k-2)/(8k-3)$; if the first two edges do not lead to A then the probability that the third does is at least $(k-3)/(8k-5)$, and so on. Hence the probability that $w \notin \Gamma_H(A)$ is at most

$$\left(1 - \frac{k-1}{8k-1}\right)\left(1 - \frac{k-2}{8k-3}\right)\cdots\left(1 - \frac{1}{6k+3}\right)$$
$$\leqslant \exp\left\{-\sum_{i=1}^{k-1} \frac{i}{6k+2i+1}\right\} \leqslant e^{-k/16} \leqslant e^{-8}.$$

Now if there is an obstruction at A, then more than $n/4$ vertices of W' do not belong to $\Gamma_H(A)$, as $|\Gamma(A)| < n/2$ and $|W'| \geqslant 3n/4$. The probability of this is at most

$$\binom{|W'|}{n/4}(e^{-8})^{n/4} \leqslant 3^{3n/4}2^{-n/2}e^{-2n} = (3^{3/4}/e^2\sqrt{2})^n < 2^{-2n}.$$

Hence the probability that some set $A \subset U$ turns out to be an obstruction is less than 2^{-n}. Finally, this shows that the probability that M constructs a complete matching in G, no matter what N does, is at least $1 - 2^{-n+1}$. □

We use this result to show that C can win the rotor game on graphs of arbitrary large girth and constant average degree. We recall the definition of the rotor game: given a (V_1, V_2, V_3) tripartite graph with $|V_1| = |V_2| = |V_3|$, the goal of C is to achieve complete oriented matchings from V_1 to V_2, from V_2 to V_3 and from V_3 to V_1, while A tries to block this.

Theorem 4.4. *Given $g \geqslant 3$, if n is sufficiently large, there is a graph G of order $3n$, size $2^{12}(3n)$ and girth at least g on which the creator C wins the rotor game.*

Proof. Let V_1, V_2 and V_3 be disjoint sets, each with n elements. Let G be the random multigraph formed as the union of three elements of $\mathscr{G}(n,n;2^{11}\text{-reg},2^{11}\text{-reg})$ a graph G_1 with bipartition (V_1,V_2), a graph G_2 with (V_2,V_3) and the third, G_3, with (V_3,V_1). This multigraph has $2^{12}(3n)$ edges.

For $l \geqslant 2$ let $X_l = X_l(G)$ be the number of l-cycles in G. Standard arguments imply that each X_l has an asymptotically Poisson distribution with bounded mean and $(X_2, X_3, \ldots, X_{g-1} = 0$ are asymptotically independent (see [6, Ch. II]). In particular $\mathbb{P}(X_2 = X_3 = \cdots = X_{g-1} = 0) \geqslant \varepsilon$ for some fixed $\varepsilon > 0$ if n is sufficiently large. Thus the probability that $G = G_1 \cup G_2 \cup G_3$ has girth at least g is at least ε. Hence by Theorem 4.3 if n is large enough then the girth of some G is at least g and M wins the complete matching game on each G_i, $i = 1,2,3$.

It is immediate that C wins the rotor game on every such G. Indeed C can win the complete matching game on G_1 so constructs a oriented complete matching from V_1 to V_2, and similarly from V_2 to V_3, and also from V_3 to V_1, thereby wins the rotor game. □

5. Generalizations

In this section we first examine the oriented cycle game played on a multigraph. Our goal is to reduce this case for playing the game on simple graphs. We also give a sufficient edge density condition for C to win, similarly to Theorem 3.3. We shall get other interesting results by letting A start instead of C or considering general sequences of A's and C's determining the order of the moves, the same way as the Strong Orientation game has been generalized in [7].

Note that if the oriented cycle game is played on a multigraph G, then the creator C wins the game whenever any of the following properties holds:

(1) G has a loop,

(2) G has an edge with multiplicity at least 3,

(3) G has a double edge and the total number of edges is odd.

The following theorem can be proved easily by noting that in case we have an even number of edges we can contract the double edges of the multigraph without changing the result of the oriented cycle game.

Theorem 5.1. *The creator C wins the oriented cycle game on a multigraph G if and only if at least one of the following conditions holds:*

- *Any of* (1), (2) *or* (3) *holds for* G.
- *Condition* (2) *holds after all double edges have been contracted in any order.*
- *After the contraction of all double edges* (*in any order*) C *wins on the resulting simple graph* H.

The analogue of Theorem 3.3 for multigraphs is the following.

Theorem 5.2. *If a multigraph G has a subgraph H of order n and size at least $2n-1$ then C wins the oriented cycle game. Furthermore if H has $2n-2$ edges and is balanced* (*i.e. no subgraph of order k has size greater than $2k-2$*) *then A wins on H if and only if H is obtained from a tree on n vertices by doubling each edge.*

Proof. As Lemmas 3.1 and 3.2 hold for multigraphs, if H has n vertices and $2n-1$ edges then by applying Lemma 3.1 for the most dense balanced subgraph $M \subset H$ we get a quasi 2-system of M. Playing in that subgraph, C wins the game with the same strategy as in Theorem 3.3.

If H has only $2n-2$ edges, the proof of Theorem 3.3 can be copied if C is able to start with a single edge (thus after two steps two points will have out/in degree at least 1). This happens unless all edges are multiple edges. C can also win easily if H is not connected, by winning on the denser component. Hence the only case when A might be the winner is when H is a double-tree. Clearly A can win on those graphs by orienting the pairs of edges chosen by C the same way as C did. □

Next we show that changing the first player might indeed change the result of the oriented cycle game.

Lemma 5.3. *In the oriented cycle game on $K_{3,3}$, whoever starts the game loses.*

Proof. We leave the proof for the reader as an easy exercise. Let us give the first steps as a hint: if C starts A should first orient an independent edge in the opposite direction as C did, if A starts C should achieve an oriented two-path with his first move. □

Clearly $G = K_{3,3} + \{\text{a pendant edge}\}$ is a graph where whoever starts the game wins. Also $H = K_{3,3} + \{2 \text{ pendant edges}\}$ is a graph where C wins on a subgraph (the subgraph isomorphic to G) but can not win on the whole graph.

Let us consider sequences of length $E(G)$ containing A's and C's next. Such a sequence (known for both A and C) will determine the order of the moves. For instance the original game corresponds to the sequence $CACAC\ldots$.

Suppose first that the game is played on the complete graph K_n. Checking all the different games determined by all the $2^{\binom{n}{2}}$ sequences we find that A wins in 6 out of the 8 cases when playing on K_3, but wins only 26 out of the 64 possible games on K_4. It is easy to show that the larger the complete graph is, the smaller the chances are that A wins, i.e.

$$\frac{\text{\# of sequences when } A \text{ wins in } K_n}{2^{\binom{n}{2}}} \to 0$$

exponentially fast as $n \to \infty$.

At last we shall examine the special sequence, when each move of C is followed by r moves of A. We consider the following question: what is the maximal value of r for which C has a winning strategy on K_n if every move of C is countered by r moves of A? Denoting this maximum by $r(n)$, Alon [2] has shown that $r(n) \geqslant \lfloor n/4 \rfloor$. The following result is obtained by a slight elaboration of Alon's argument.

Theorem 5.4. *For $n \geqslant 3$, we have $r(n) \geqslant \lfloor (2-\sqrt{3})n \rfloor$: if A replies by $r = \lfloor (2-\sqrt{3})n \rfloor$ moves to every move of C then the creator wins the oriented cycle game on K_n.*

Proof. Let $\vec{G}_k$ be the oriented graph arising after the moves have been made in the kth round, so that $\vec{G}_0$ is the empty graph on V, $|V| = n$, and $e(\vec{G}_k) = k(r+1)$. For $k \geqslant 0$ and $v \in V$, set

$$S_k(v) = \{w \in V : \vec{G}_k \text{ contains an oriented path from } w \text{ to } v\},$$

$$T_k(v) = \{w \in V : \vec{G}_k \text{ contains an oriented path from } v \text{ to } w\},$$

$$m(k) = \max\{|S_k(v) \cup T_k(v)| : v \in V\}.$$

Note that C wins the game if he can achieve that for some k and v, the set $S_k(v) \cap T_k(v)$ contains at least one vertex other than v.

To simplify our calculations let $s = \lfloor r/4 \rfloor + 2$. The creator plays the game in three phases. First he simply plays to maximize $m(k)$ until $m(k) \geqslant k+s$, then he fixes v and tries to balance $|S_k(v)|$ and $|T_k(v)|$, and in the third phase he increases $|S_k(v) \cup T_k(v)|$ keeping the two sets as balanced as possible.

Phase 1: We claim that for $k \leqslant s$ we can guarantee that $m(k) \geqslant 2k$. Clearly $m(1) \geqslant 2$, suppose that $k \leqslant s-1$ is the last time we have $m(k) \geqslant 2k$. Then $m(k) = 2k$, $\Delta(\vec{G}_k) \leqslant 2k-1$ and every arc of $\vec{G}_k$ has at least one endvertex in $S_v(k) \cup T_v(k)$. Here

$$k(r+1) = e(\vec{G}_k) \leqslant 2k(2k-1) - (2k-1),$$

contradicting $k \leqslant s-1$.

Thus C can reach $m(i) \geqslant i+s$ for some $i \leqslant s$ and $v \in V$. We fix this v to be the central vertex from now on.

Phase 2: Without loss of generality $|S_i(v)| \leqslant |T_i(v)|$ and then C will increase $|S_k(v)|$ by at least one with each move, until it is at least as large as $\lfloor (k+s+1)/2 \rfloor$. It is easily checked that the creator can make these moves by orienting edges from vertices not in $S_k(v) \cup T_k(v)$ to the center v.

Phase 3: In this phase with each move C increases one of $|S_k(v)|$ and $|T_k(v)|$, provided it is not strictly larger than the other, until the round $k = n-s+1$ when

$|S_k(v)| + |T_k(v)| \geqslant n+2$ ensures $|S_k(v) \cap T_k(v)| \geqslant 2$. All we have to check is that the creator can achieve this. Indeed, if this is not the case then C cannot make an appropriate move after $\vec{G}_k$ has been constructed. Then, assuming that $|S_k(v)| \leqslant |T_k(v)|$, say, we have $|S_k(v)| \geqslant \lfloor (k+s+1)/2 \rfloor$, so

$$k(r+1) = e(\vec{G}_k) \geqslant \lfloor (k+s+1)/2 \rfloor (n - \lfloor (k+s+1)/2 \rfloor),$$

which can easily be checked to contradict the definitions of r and s, completing the proof. □

The proof can easily be modified to work even if A is allowed to reply by no more than $r = \lfloor (2-\sqrt{3})n \rfloor$ moves to every move of C.

Concerning the upper bound for $r(n)$, it is clear that $r(n) \leqslant n-3$, i.e. A is the winner if he orients $r \geqslant n-2$ edges after each move of C. In fact we conjecture that $r(n) = n-3$ for $n \geqslant 3$, which can easily be checked for $n = 3,4,5,6$.

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